Introduction to axial algebras

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Contents

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1 Motivation and Background

We begin with some motivation/background/history for axial algebras.

- **Vertex operator algebras** (VOAs) were first considered by physicists in connection with chiral algebras and 2D conformal field theory. Mathematicians became interested in them through the links with Monstrous moonshine [8]. The moonshine VOA $V^\natural$ (or sometimes $V^\sharp$ for those who prefer a brighter tone) was instrumental in Borcherd's proof [1], for which he won a Field's Medal. (Very) Roughly speaking, a VOA is an infinite dimensional graded vector space $V = \bigoplus_{i \in \mathbb{Z}^{\geq 0}} V_i$, where the V_i are finite dimensional, with infinitely many products which are linked in an intricate way. The moonshine VOA $V^\natural$ has the 196,883-dimensional Griess algebra as its weight two component and both the Griess algebra and $V^\natural$ have the Monster as their automorphism group.

- **Majorana algebras** are the predecessors of axial algebras. They were introduced by Sasha Ivanov [13] to axiomatise some key properties of $V^\natural$. Norton showed for the Griess algebra [2], and later Miyamoto for (OZ-type) VOAs [18], that there are idempotents called *axes*, respectively *Ising vectors*, which are in bijection with involutions of the algebra. For the Griess algebra, these involutions are the class of 2A involutions which generate the Monster; correspondingly, the axes generate the Griess algebra. Analogous statements hold for $V^\natural$. Majorana algebras were introduced to axiomatise these properties and cover subalgebras of the Griess algebra (and some others), whereas axial algebras are a wider class of algebra.

- **Jordan algebras** were introduced in 1933 to study observables in quantum mechanics. A *Jordan algebra* is a commutative non-associative algebra which satisfies

$$(xy)(xx) = x(y(xx))$$

for all x and y. It is known that idempotents in a Jordan algebra have eigenvalues 1, 0 and $\frac{1}{2}$. It turns out that, when there are enough idempotents to generate the algebra, Jordan algebras are examples of axial algebras.

- **Matsuo algebras** are non-associative algebras defined from Fischer spaces, which in turn can be defined from a 3-transposition group (that is, a group generated by involutions such that $|ab| \leq 3$ for all generating involution a and b). Conversely, given a 3-transposition group, it defines a Fischer space and a Matsuo algebra. Examples of 3-transposition groups include symplectic, unitary and orthogonal groups in characteristic two and also the Fischer groups Fi_{22}, Fi_{23} and Fi_{24}. All Matsuo algebras are axial algebras.

- **Finite simple groups** Axial algebras are designed to have a link between idempotents in the algebra and involutions generating a group. Jordan algebras and Matsuo algebras are among the simplest axial algebras, called *Jordan type*, and have related to them all 3-transposition groups and some classical groups, F_4 and G_2. The next simplest type is called (*generalised*) *Monster type* and has many related groups, including the Monster. There are also axial algebras associated to some groups of Lie type with a simply-laced diagram [4]. So axial algebras are allowing us to view all these different simple groups in the same framework, potentially providing a new way to view the Classification of Finite Simple Groups.

- **PDEs** Recently Tkachev has found examples of axial algebras which occur in the theory of non-linear PDEs, namely Hsiang algebras [24]. An idempotent here has eigenvalues 1, -1, $-\frac{1}{2}$ and $\frac{1}{2}$. In fact, Tkachev shows that if A is any commutative, non-associative algebra over a field of characteristic not 2, or 3 which satisfies a non-trivial polynomial $P(z)$ in one variable, then any idempotent x has $\frac{1}{2}$ as an eigenvalue of ad_x. Moreover, the multiplication of the eigenvectors (fusion law) is restricted.

- **Algebras of vector flows on manifolds** Axial type behaviour has also been speculated by Fox in algebras of vector flows on manifolds, such as Ricci flow [5].

Axial algebras were introduced by Hall, Rehren and Shpectorov in [9]. The material for these notes has been collected from there and a number of other sources including [16, 23].

2 Axial algebras

Throughout, let $\mathbb{F}$ be a field; we place no restriction on the characteristic yet. (Most of our definitions also hold with a ring, but we restrict ourselves to a field here.) An *algebra* is a vector space over $\mathbb{F}$ with a multiplication $\cdot\colon A \times A \to A$ which distributes over addition. We do not assume that our algebra has an identity element, or that there are multiplicative inverses. In fact, the algebras we will consider will almost never have an identity. They will be commutative, but non-associative, by which we mean that they are not necessarily associative. That is, in general

$$x(yz) \neq (xy)z$$

for x, y, z in the algebra.

In non-associative algebras, we need some extra structure to make up for the lack of associativity of the product. In many classes of algebra this is provided by requiring that all elements of the algebra satisfies one or more identities. For example, elements in a Jordan algebra satisfy the identity $(xy)(xx) = x(y(xx))$ and in Lie algebras satisfy alternativity and the Jacobi identity.

In axial algebras, we do not require that the algebra satisfies an identity for all elements. Instead we have a set of distinguished elements, called axes. We require these to be semisimple and furthermore we restrict the multiplication of their eigenvectors (with respect to the adjoint action). This is done via a fusion law.

2.1 Fusion law

Before we define axial algebras, we first need to describe a fusion law.

Definition 2.1. Let $\mathcal{F}$ be a set and $\star\colon \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ be a symmetric binary operation. We call the pair $\mathcal{F} = (\mathcal{F}, \star)$ a *fusion law*[1]

Since we can always extend $\star$ to subsets of $\mathcal{F}$, we will often abuse the notation in this way and write $S \star T$ for $\bigcup_{s\in S, t\in T} s \star t$.

Since $\star$ is a binary operation, we will normally represent $\mathcal{F}$ as a table. To make these tables neater, we will usually leave out the set brackets and just write $1, 0$ for the set $\{1, 0\}$, for example. We also just leave a blank instead of writing the empty set. Table 1 shows some common fusion laws which we will meet later. We call these the *associative fusion law* $\mathcal{A}$, the fusion law of *Jordan type* η $\mathcal{J}(\eta)$ and the (*generalised*) *Monster fusion law* $\mathcal{M}(\alpha, \beta)$.

	1	0
1	1	
0		0

	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 1: Fusion laws $\mathcal{A}$, $\mathcal{J}(\eta)$, and $\mathcal{M}(\alpha, \beta)$

We will return to some features of these tables later.

[1]Note that in some of the older papers, this was referred to as fusion rules, but this led to some awkward singular/plural issues.

2.2 Axes and axial algebras

Let A be a commutative, non-associative algebra. Recall that, for an element $a \in A$, the *adjoint endomorphism* $\mathrm{ad}_a \colon A \to A$ is defined by $v \mapsto av$. (Note that since the algebra is commutative, we do not have to worry whether this is the left, or right adjoint.) Let $\mathrm{Spec}(a)$ be the set of eigenvalues of ad_a. For $\lambda \in \mathbb{F}$, let $A_\lambda(a)$ be the λ-eigenspace of ad_a (in particular, $A_\lambda(a)$ may be 0). Where the context is clear, we will write A_λ for $A_\lambda(a)$. If $S \subseteq \mathbb{F}$, then we will also write A_S for $\bigoplus_{\lambda \in S} A_\lambda$.

We say that $a \in A$ is *semisimple* if the adjoint ad_a is diagonalisable. This is equivalent to the algebra decomposing as the direct sum of eigenspaces for ad_a:

$$A = A_{\mathrm{Spec}(a)} := \bigoplus_{\lambda \in \mathrm{Spec}(a)} A_\lambda(a)$$

Definition 2.2. Let $\mathcal{F} = (\mathcal{F}, \star)$ be a fusion law. An element $a \in A$ is an *$\mathcal{F}$-axis* if the following hold:

1. a is an idempotent (i.e. $a^2 = a$)
2. a is semisimple and $A = A_\mathcal{F}$
3. $A_\lambda A_\mu \subseteq A_{\lambda \star \mu}$, for all $\lambda, \mu \in \mathcal{F}$

We say that an axis is *primitive* if $A_1 = \langle a \rangle$.

Note that $A = A_\mathcal{F}$ implies that $\mathrm{Spec}(a) \subseteq \mathcal{F}$. When the fusion law is clear from context, we will just call a an axis. We will almost always assume that an axis is primitive (we will make clear when this is not the case).

Note that, as a is an idempotent, we always have that $a \in A_1$. An axis is primitive if (the span of) a is all of the 1-eigenspace. This notion of primitivity generalises the usual definition. Indeed, if an idempotent a has a direct sum decomposition $a = a_1 + \dots + a_n$ where the a_i are idempotents and $a_i a_j = 0$ for $i \neq j$, we usually say it is primitive if $a = a_i$ and $a_j = 0$ for all $j \neq i$. In our definition,

$$aa_i = (a_1 + \dots + a_n)a_i = a_i^2 = a_i$$

and so a_i is a 1-eigenvector for all i. Since we assume the 1-eigenspace is 1-dimensional, a_i is a scalar multiple of a_j and so a is primitive in the usual sense.

Definition 2.3. An *$\mathcal{F}$-axial algebra* is a pair (A, X) of a commutative non-associative algebra A and a set of $\mathcal{F}$-axes X which generate A. We say A is *primitive* if all the axes in X are primitive.

As with axes, we will almost always only consider primitive axial algebras. We will also frequently drop the fusion law where it is understood, and just refer to A as an axial algebra.

It is important to note that even though we will often just write A instead of (A, X), an axial algebra always has a distinguished set X of generators.

Remark 2.4.

1. All the axes in the generating set X satisfy the same fusion law $\mathcal{F}$.

2. Although two axes $a, b \in X$ have the same fusion law, we do not assume that the dimensions of their corresponding eigenspaces are the same. So, $\dim(A_\lambda(a))$ does not necessarily equal $\dim(A_\lambda(b))$.

3. Moreover, for an axis $a \in X$ and $\lambda \in \mathcal{F}$, we do not require that $A_\lambda(a)$ is not the zero subspace. So, in particular, an $\mathcal{A}$-axial algebra is a $\mathcal{J}(\eta)$-axial algebra for all $\eta \neq 1, 0$ and also an $\mathcal{M}(\alpha, \beta)$-axial algebra for all $\alpha \neq \beta \in \mathbb{F} - \{1, 0\}$. Likewise, a $\mathcal{J}(\eta)$-axial algebra is an $\mathcal{M}(\eta, \beta)$-axial algebra for all $\beta \in \mathbb{F} - \{1, 0, \eta\}$. It is also a $\mathcal{M}(\alpha, \eta)$-axial algebra for all $\alpha \in \mathbb{F} - \{1, 0, \eta\}$ since $\eta \star \eta = \{1, 0\} \subset \{1, 0, \alpha\}$.

Before we go on, let us see some examples.

Example 2.5. The motivating example for axial algebras is the Griess algebra. It is a $196,884$-dimensional non-associative algebra over $\mathbb{R}$. Norton showed that it contains idempotents, that he calls axes, which generate the algebra [2]. These axes have eigenvalues 1, 0, $\frac{1}{4}$ and $\frac{1}{32}$ and it turns out that these satisfy the fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$. Hence the Griess algebra is an axial algebra. It is for this reason we call the fusion law the *Monster fusion law*, or the fusion law of *Monster type*.

	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Table 2: Monster fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$

Example 2.6. Let A be an axial algebra generated by two axes a and b such that $ab = 0$. Clearly, A is spanned by a and b, so it is 2-dimensional, and it also has fusion law $\mathcal{A}$, hence it is an axial algebra. For reasons which may appear mysterious[2], this algebra is known as 2B. However, we are all very familiar with it – A is in fact associative and so $A \cong \mathbb{F} \oplus \mathbb{F}$. We will see later that any $\mathcal{A}$-axial algebra is associative and is the direct sum of copies of the field.

Having given you one example you can't work with and another which isn't very interesting, we now see an interesting family of examples.

Example 2.7 (Matsuo algebras)**.** Let (G, D) be a 3-transposition group. (So, D is a set of involutions which is G-invariant, generates G and such that $o(ab) \leq 3$ for all $a, b \in D$.) Let $a, b \in D$. Then, $o(ab) = 3$ if and only if they generate a dihedral group D_6 of order 6. In particular, there is one other involution $c \in D_6 \cong S_3$ not equal to a or b. Since D is G-invariant, $c \in D$.

Let $M_\eta(G)$ be the algebra with basis D and multiplication given by

$$ab = \begin{cases} a & \text{if } a = b \\ 0 & \text{if } o(ab) = 2 \\ \frac{\eta}{2}(a + b - c) & \text{if } o(ab) = 3 \end{cases}$$

for some $\eta \in \mathbb{F} \setminus \{1, 0\}$. Then, it turns out that A is a primitive $\mathcal{J}(\eta)$-axial algebra.

Exercise 2.8. For the group S_3 with D being the three involutions, consider $M_\eta(S_3)$. Find the eigenspaces associated to an axis and verify that they do indeed satisfy the $\mathcal{J}(\eta)$ fusion law. We often call this algebra 3C(η)[3].

3 Automorphisms

One of the key features of axial algebras is that there is a naturally associated group of automorphisms. In fact, we will associate automorphism(s) to each axis. For the motivating example of the Griess algebra, or $V^\natural$, we have a 2A involution from the Monster associated to each axis. Moreover, the class of 2A involutions generates the Monster.

The way we associate automorphisms to the axes relies on the fusion law being graded.

[2]This naming comes from the Griess algebra. Recall from the introduction that axes in the Griess algebra are in bijection with 2A-involutions in the Monster. We label 2-generated subalgebras by the conjugacy class of the product of the two 2A involutions associated to the generators. One can show that this gives a well-defined label. There are eight types 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A and 6A.

[3]This is a generalised version of the 3C subalgebras of the Griess algebra.

3.1 Morphisms and gradings of fusion laws

Here we follow the more recent definition of gradings which comes from De Medts, Peacock, Shpectorov and Van Couwenberghe [3]. It is equivalent to the one given in [16], but allows us to endow fusion laws with more structure.

Definition 3.1. Let $\mathcal{F}_1 = (\mathcal{F}_1, \star_1)$ and $\mathcal{F}_2 = (\mathcal{F}_2, \star_2)$ be two fusion laws. A *morphism* of fusion laws is a map $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ such that

$$f(\lambda \star_1 \mu) \subseteq f(\lambda) \star_2 f(\mu)$$

for all $\lambda, \mu \in \mathcal{F}_1$.

Note that this allows us to define the category of fusion laws, with objects being fusion laws and morphisms in the category being morphisms in the above sense. More details about this category can be found in [3].

In order for us to define a grading, we first consider another example of a fusion law.

Example 3.2. Let T be an abelian group. Then we can define the *group fusion law* as $(T, \star)$, where

$$s \star t = \{st\}$$

for all $s, t \in T$.

It is easy to see that the category of abelian groups form a full subcategory of the category of fusion laws. That is, the morphisms between two abelian groups are precisely the morphism between the two group fusion laws.

Definition 3.3. Suppose T is an abelian group. A *T-grading* of the fusion law $\mathcal{F}$ is a morphism f from $\mathcal{F}$ to the group fusion law T. We say the grading is *adequate* if $f(\mathcal{F})$ generates T.

Since any non-adequate grading can always be restricted to an adequate grading, from now on we will always consider adequate gradings and just talk of a grading.

We will be most interested in C_2-gradings. Here, we will use the convention that $C_2 = \{+, -\}$.

Example 3.4. The fusion laws $\mathcal{A}$, $\mathcal{J}(\eta)$ for $\eta \in \mathbb{F} - \{1, 0\}$, $\mathcal{M}(\alpha, \beta)$ for $\alpha, \beta \in \mathbb{F} - \{1, 0\}$, $\alpha \neq \beta$, are all C_2-graded fusion laws.

It will often be convenient to think of the t-graded piece of $\mathcal{F}$, for $t \in T$, by which we mean the full preimage $f^{-1}(t) \subseteq \mathcal{F}$. Note that these graded pieces partition $\mathcal{F}$ – this is in the spirit of the original definition of grading. In $\mathcal{M}(\alpha, \beta)$, the positively graded piece is $\{1, 0, \alpha\}$ and the negatively graded piece is $\{\beta\}$.

Let A be an algebra and $a \in A$ an $\mathcal{F}$-axis (note that we do not require A to be an axial algebra). If f is a T-grading of $\mathcal{F}$, then this induces a natural T*-grading* on A with respect to the the axis a by considering the full preimage. Namely, we will write

$$A_t = \bigoplus_{\lambda \in f^{-1}(t)} A_\lambda$$

3.2 Automorphisms

When $\mathcal{F}$ is T-graded, this leads to automorphisms of the algebra. Let T^* be the group of linear characters of T over $\mathbb{F}$. That is, the set of all homomorphisms from T to the multiplicative group $\mathbb{F}^\times$. For an axis $a \in X$ and $\chi \in T^*$, consider the linear map $\tau_a(\chi)\colon A \to A$ defined by

$$u \mapsto \chi(t)u \qquad \text{for } u \in A_t(a)$$

and extended linearly to A.

Lemma 3.5. *The map $\tau_a(\chi)$ is an automorphism of A. Furthermore, the map sending χ to $\tau_a(\chi)$ is a homomorphism from T^* to* $\mathrm{Aut}(A)$*.*

Proof. Note that on A_λ the map $\tau_a(\chi)$ just acts as scalar multiplication by $\chi(t)$. So the second part follows immediately. For the first, it suffices to check the multiplication on the graded parts. Let $x \in A_t$, $y \in A_s$. Since A is T-graded,

$$\tau_a(\chi)(xy) = \chi(ts)xy = \chi(t)\chi(s)xy = \tau_a(\chi)(x)\ \tau_a(\chi)(y)$$

and so $\tau_a(\chi)$ is an automorphism of A. $\square$

We call $\tau_a(\chi)$ a *Miyamoto automorphism.*

Definition 3.6. We call the image T_a of the map $\chi \mapsto \tau_a(\chi)$, the *axis subgroup* of $\mathrm{Aut}(A)$ corresponding to a.

Usually, T_a is a copy of T^*, but occasionally, when some subspaces $A_t(a)$ are trivial, T_a could be isomorphic to a factor group of T^* over a nontrivial subgroup.

Also, note that T^* is isomorphic to some quotient of T. We will mostly be interested in the case when $T \cong T^*$. For this to be the case, we cannot have the characteristic of the field dividing $|T|$ and we also must have the field large enough so that it contains all the relevant roots of unity.

In any case, we are most interested in the case where $T = C_2$. Here, we require that $\mathrm{char}(\mathbb{F}) \neq 2$. Then, since the square root of unity -1 is always in the field, $T^* \cong T$. In this case there are just two linear characters, the trivial character χ_1 and the sign character χ_{-1}. Since the trivial character always leads to the trivial automorphism, we are left with just one non-trivial automorphism for a, $\tau_a(\chi_{-1})$. To simplify notation, we will write this as $\tau_a = \tau_a(\chi_{-1})$ and call it a *Miyamoto involution* (as it has order 2). Here $T_a = \langle \tau_a \rangle \cong C_2$.

As we previously noted, the Monster fusion law is C_2-graded. So for example of the Griess algebra, we may associate a Miyamoto involution τ_a to each axis a. These are indeed the same involutions as Norton found, so we have successfully generalised the situation found in the Griess algebra.

Recall now that every axial algebra A comes with a set of generating axes X. In the following definition we slightly relax conditions on X by allowing it to be an arbitrary set of axes from A.

Definition 3.7. The *Miyamoto group* $\mathrm{Miy}(X)$ of A with respect to the set of axes X is the subgroup of $\mathrm{Aut}(A)$ generated by the axis subgroups T_a, $a \in X$.

As noted before, in the Griess algebra A, the Miyamoto involutions are involutions in the Monster in the conjugacy class 2A. Since one can show that this class geneerates the Monster, the Miyamoto group of the Griess algebra is indeed the Monster.

Note that in this case, it turns out that the map $a \mapsto \tau_a$ is a bijection. However, this does not need to be the case for other axial algebras. Indeed, if we consider $2\mathrm{B} = \langle a, b \rangle$ to be a $\mathcal{M}(\alpha, \beta)$-axial algebra, then since the β-eigenspace of each axis is trivial, $\tau_a = \tau_b = 1$ and so the τ map is not bijective. There are more complicated examples where two axes have non-trivial, but equal Miyamoto involutions.

3.3 Closed sets of axes

On the face of it, our definition of the Miyamoto group depends on the choice of axes. It is also possible that two different sets of axes can generate axial algebras which are isomorphic as algebras. In this section we will iron out some of these difficulties.

First, note that if a is an axis and $g \in \mathrm{Aut}(A)$, then a^g is again an axis. We record this in the following lemma.

Lemma 3.8. *Let a be an $\mathcal{F}$-axis in an algebra A and $g \in \mathrm{Aut}(A)$. Then, a^g is also an $\mathcal{F}$-axis with $A_\lambda(a^g) = A_\lambda(a)^g$. Moreover, $\tau_{a^g}(\chi) = \tau_a(\chi)^g$ and hence $T_{a^g} = T_a^g$.*

Proof. It is clear that a^g is an idempotent and $A_\lambda(a^g) = A_\lambda(a)^g$. Hence, for $\lambda, \mu \in \mathcal{F}$, we have

$$A_\lambda(a^g)A_\mu(a^g) = A_\lambda(a)^g A_\mu(a)^g = (A_\lambda(a)A_\mu(a))^g \subseteq A_{\lambda\star\mu}(a)^g = A_{\lambda\star\mu}(a^g)$$

Since the $\tau_a(\chi)$ maps are defined as scalar multiplication on the eigenspaces, it is clear that $\tau_{a^g}(\chi) = \tau_a(\chi)^g$ and hence $T_{a^g} = T_a^g$. □

Now that we know how automorphisms act on axes, we can define the closure of a set of axes in the natural way.

Definition 3.9. A set of axes X is *closed* if it is closed under the action of its Miyamoto group $\mathrm{Miy}(X)$. That is, $X^{\mathrm{Miy}(X)} = X$. Equivalently, $X^\tau = X$ for all $\tau \in T_a$ with $a \in X$.

It is easy to see that the intersection of closed sets is again closed and so every X is contained in the unique smallest closed set $\bar{X}$ of axes. We call $\bar{X}$ the *closure* of X.

Lemma 3.10. *For a set of axes X, we have that $\bar{X} = X^{\mathrm{Miy}(X)}$ and furthermore $\mathrm{Miy}(\bar{X}) = \mathrm{Miy}(X)$.*

Proof. Since $X \subseteq \bar{X}$, we have that $\mathrm{Miy}(X) \leq \mathrm{Miy}(\bar{X})$. Hence $X^{\mathrm{Miy}(X)} \subseteq \bar{X}^{\mathrm{Miy}(\bar{X})} = \bar{X}$. To show the reverse inclusion, it suffices to prove that $X^{\mathrm{Miy}(X)}$ is closed.

We claim that $\mathrm{Miy}(X^{\mathrm{Miy}(X)}) = \mathrm{Miy}(X)$. Suppose that $b \in X^{\mathrm{Miy}(X)}$. Then, $b = a^g$ for some $a \in X$ and $g \in \mathrm{Miy}(X)$. By Lemma 3.8, $T_b = T_{a^g} = T_a^g$. Since $T_a \leq \mathrm{Miy}(X)$ and $g \in \mathrm{Miy}(X)$, we have that $T_b = T_a^g \leq \mathrm{Miy}(X)^g = \mathrm{Miy}(X)$. Hence, $\mathrm{Miy}(X^{\mathrm{Miy}(X)}) = \mathrm{Miy}(X)$ as claimed. Clearly, $X^{\mathrm{Miy}(X)}$ is invariant under $\mathrm{Miy}(X) = \mathrm{Miy}(X^{\mathrm{Miy}(X)})$, hence $X^{\mathrm{Miy}(X)}$ is closed. Finally, since $\bar{X} = X^{\mathrm{Miy}(X)}$, $\mathrm{Miy}(\bar{X}) = \mathrm{Miy}(X^{\mathrm{Miy}(X)}) = \mathrm{Miy}(X)$. $\square$

Turning again to the example of the Griess algebra, it is well-known that the Monster M can be generated by three 2A involutions, say, τ_a, τ_b, and τ_c. Since the 2A involutions are in bijection with the axes, we may suppose that the corresponding axes are $a, b, c \in A$. Setting $X = \{a, b, c\}$, we have that $\mathrm{Miy}(X) = \langle T_a, T_b, T_c \rangle = \langle \tau_a, \tau_b, \tau_c \rangle = M$. Hence $\bar{X} = X^{\mathrm{Miy}(X)} = X^M$ is the set of all axes of A, since $\{\tau_a, \tau_b, \tau_c\}^M$ is clearly all of the 2A conjugacy class. (We again use the fact that the map sending an axis to the corresponding 2A involution is bijective.) So here $\bar{X}$ (of size approximately 9.7×10^{19}) is huge compared to the tiny X.

We have seen that different sets of axes can generate the same axial algebra and, crucially, also give the same Miyamoto group. This suggests the following definition.

Definition 3.11. We say that sets X and Y of axes are *equivalent* (denoted $X \sim Y$) if $\bar{X} = \bar{Y}$.

Clearly, this is indeed an equivalence relation on sets of axes. Furthermore, for equivalent sets, we have that $\mathrm{Miy}(X) = \mathrm{Miy}(\bar{X}) = \mathrm{Miy}(\bar{Y}) = \mathrm{Miy}(Y)$, so their Miyamoto groups are the same. Since $\bar{X} = X^{\mathrm{Miy}(X)}$ and, similarly, $\bar{Y} = Y^{\mathrm{Miy}(Y)}$, we can also state the following.

Lemma 3.12. *Sets X and Y of axes are equivalent if and only if both the following two conditions hold:*

1. *$G := \mathrm{Miy}(X) = \mathrm{Miy}(Y)$, and*

2. *Every $x \in X$ is G-conjugate to some $y \in Y$ and, vice versa, every $y \in Y$ is G-conjugate to some $x \in X$.* □

Now that we have a notion of equivalence of axes, we have the following natural definition.

Definition 3.13. A property of axial algebras is called *stable* if it is invariant under equivalence of axes.

We can now reword the second part of Lemma 3.10.

Corollary 3.14. *The Miyamoto group of an axial algebra is stable.*

3.4 Invariance

Let $a \in X$ be an axis and W be a (vector) subspace of A invariant under the action of ad_a. Since ad_a is semisimple on A, it is also semisimple on W, and so

$$W = \bigoplus_{\lambda \in \mathcal{F}} W_\lambda(a)$$

where $W_\lambda(a) = W \cap A_\lambda(a) = \{w \in W : aw = \lambda w\}$.

Let us note the following important property of axis subgroups T_a.

Lemma 3.15. *For an axis a, if a subspace $W \subseteq A$ is invariant under ad_a then W is invariant under every $\tau_a(\chi)$, $\chi \in T^*$. (That is, W is invariant under the whole T_a.)*

Proof. We have just seen that $W = \bigoplus_{\lambda \in \mathcal{F}} W_\lambda(a)$ where $W_\lambda(a)$ is a subspace of $A_\lambda(a)$. Since $\tau = \tau_a(\chi)$ acts on $A_\lambda(a)$ as a scalar transformation, it leaves invariant every subspace of $A_\lambda(a)$. In particular, $W_\lambda(a)^\tau = W_\lambda(a)$ for every λ, and so $W^\tau = W$. □

For example, ideals of A are invariant under ad_a for all axes a. Hence we have the following which will be useful later.

Corollary 3.16. *Every ideal I of A is $\mathrm{Miy}(X)$-invariant for any set of axes X in A.*

Let us now prove the following important property. We denote by $\langle\langle Y \rangle\rangle$ the subalgebra of A generated by the set Y.

Theorem 3.17. *Suppose that $X \sim Y$. Then $\langle\langle X \rangle\rangle = \langle\langle Y \rangle\rangle$. In particular, if X generates A then so does Y. Hence generation of an axial algebra is stable.*

Proof. Let $B = \langle\langle X \rangle\rangle$ and $C = \langle\langle Y \rangle\rangle$. Note that B is invariant under ad_a for every $a \in X$. So, by Lemma 3.15, B is $\mathrm{Miy}(X)$-invariant. Clearly, $\bar{X} = X^{\mathrm{Miy}(X)} \subseteq B$. So, $Y \subseteq \bar{Y} = \bar{X} \subseteq B$, and hence $C \subseteq B$. By symmetry, we also have $B \subseteq C$, therefore $B = C$. $\square$

So it is indeed the case that equivalent set of axes generate the same axial algebra and also give the same Miyamoto group.

We note that the converse of the above theorem does not hold. That is, there exist sets of axes X and Y which are inequivalent, but which both generate the same axial algebra A. We know of examples where we have two closed sets of axes X and Y where $X \subsetneq Y$. Both generate the same axial algebra and also $\mathrm{Miy}(X) = \mathrm{Miy}(Y)$.

The group S_4 has 6 single transpositions and 3 double transpositions. In [17, Table 4], Shpectorov and I use an algorithm to build a 9-dimensional axial algebra A (known by the shape 4B3C2A) with Miyamoto group S_4 and a set of nine axes X which are in bijection with the involutions of S_4. It turns out that the set Y of six axes which are in bijection with the single transpositions also generate the whole algebra A and $\mathrm{Miy}(Y) = S_4$ too (now known by the shape 3C2A). But clearly Y is closed under the action of the Miyamoto group $\mathrm{Miy}(Y) = S_4$. So here we have an example of two inequivalent sets of axes X and Y, but where $\mathrm{Miy}(X) = \mathrm{Miy}(Y)$. This shows that the second condition in Lemma 3.12 is indeed necessary.

Example 3.18. Let A be the Griess algebra and a, b, and c be axes such that $M = \langle \tau_a, \tau_b, \tau_c \rangle$. As noted before, the closure of $X = \{a, b, c\}$ is the set of all axes in A. Since $X \sim \bar{X}$, by Theorem 3.17, $\langle\langle X \rangle\rangle = A$. So, despite its large dimension, A can be generated by just three axes.

4 Frobenius form

VOAs and the Griess algebra both admit a bilinear form which behaves well with respect to the multiplication in the algebra. We can also have such a form.

Definition 4.1. A *Frobenius* form on an axial algebra A is a (non-zero) bilinear form $(\cdot, \cdot) \colon A \times A \to \mathbb{F}$ which associates with the algebra product. That is,

$$(a, bc) = (ab, c) \qquad \text{for all } a, b, c \in A$$

In some papers, there is also a condition on the value of (a, a) for axes $a \in X$. However, we begin by not making any such restriction. We can still prove several nice properties.

Lemma 4.2. *A Frobenius form on an axial algebra is symmetric.*

Proof. Since axial algebras are spanned by products of axes, it is enough just to consider these. Let $a, b \in A$ be products of axes. We can write $a = a_1a_2$ where a_1 and a_2 are both products of axes (if a is itself an axis, then $a = aa$). Now

$$(a, b) = (a_1a_2, b) = (a_1, a_2b) = (a_1, ba_2) = (a_1b, a_2) = (b, a_1a_2) = (b, a) \quad \square$$

The form behaves particularly well with respect to the decomposition given by an axis.

Lemma 4.3. *For an axis a, the direct sum decomposition $A = \bigoplus_{\lambda\in\mathcal{F}} A_\lambda(a)$ is orthogonal with respect to every Frobenius form $(\cdot,\cdot)$ on A.*

Proof. Suppose $u \in A_\lambda(a)$ and $v \in A_\mu(a)$ for $\lambda \neq \mu$. Then $\lambda(u, v) = (\lambda u, v) = (au, v) = (ua, v) = (u, av) = (u, \mu v) = \mu(u, v)$. Since $\lambda \neq \mu$, we conclude that $(u, v) = 0$. $\square$

There is also a partial converse to this.

Lemma 4.4. *Let A be an axial algebra which is spanned by axes X. If $(\cdot,\cdot)$ is a bilinear form on A such that*

$$A_\lambda(a) \perp A_\mu(a)$$

for all $\lambda \neq \mu$ and $a \in X$, then $(\cdot,\cdot)$ is a Frobenius form.

Proof. Let $u \in A_\lambda$, $v \in A_\mu$. If $\lambda \neq \mu$, then $(u, v) = 0$ and so

$$(ua, v) = \lambda(u, v) = 0 = \mu(u, v) = (u, av)$$

If $\lambda = \mu$, then $(ua, v) = \lambda(u, v) = (u, av)$ anyway. By bilinearity, we have that $(ua, v) = (u, av)$ for all $u, v \in A$ and $a \in X$. Since the axes X span A, the result follows from bilinearity. $\square$

Let a be a primitive axis. Then we may decompose $u \in A$ with respect to a as $u = \sum_{\lambda\in\mathcal{F}} u_\lambda$, where $u_\lambda \in A_\lambda(a)$. We call u_λ the *projection* of u onto $A_\lambda(A)$. Focusing on the projection u_1, as a is primitive, $u_1 = \varphi_a(u)a$ for some $\varphi_a(u)$ in $\mathbb{F}$. It is easy to see that φ_a is linear in u.

Proposition 4.5. *Let $(\cdot,\cdot)$ be a Frobenius form on a primitive axial algebra A. Then,*

1. $(a, u) = \varphi_a(u)(a, a)$ *for an axis $a \in X$ and $u \in A$.*

2. $(\cdot,\cdot)$ *is uniquely defined by the values (a, a) on the axes $a \in X$.*

3. $(\cdot,\cdot)$ *is invariant under the action of* $\mathrm{Miy}(X)$ *if and only if $(a, a) = (a^g, a^g)$ for all $a \in X$ and $g \in \mathrm{Miy}(X)$.*

Proof. We decompose $u = \sum_{\lambda \in \mathcal{F}} u_\lambda$ with respect to a, where $u_\lambda \in A_\lambda(a)$. Now, by Lemma 4.3, $(a, u) = (a, \sum_{\lambda \in \mathcal{F}} u_\lambda) = (a, u_1) = \varphi_a(u)(a, a)$.

For the second part, since A is an axial algebra, it is spanned by products of the axes. So, it suffices to show that the value of (w, v) is uniquely defined by the value on the axes, where w and v are products of axes.

We proceed by induction on the length of w. By the first part, if w has length one, then (w, v) is determined by the value of (w, w). Suppose that w has length at least two. Then we may write $w = w_1 w_2$ where w_1 and w_2 are both products of axes of length strictly less than w. Since the form is Frobenius, $(w, v) = (w_1 w_2, v) = (w_1, w_2 v)$. So, by induction, the form is determined by the values of (a, a) for axes $a \in X$.

Finally, for the third part, one direction is clear. So, assume that $(a, a) = (a^g, a^g)$ for all $a \in X$, $g \in \mathrm{Miy}(X)$. Again, in order to show that the form is G-invariant, it is enough to show it on products of axes w and v. Using the above argument for the second part as an algorithm, we see that there exists $a \in X$, $u \in A$, such that $(w, v) = (a, u)$. So, we also have $(w^g, v^g) = (a^g, u^g)$. Since $(a, u) = \varphi_a(u)(a, a)$, it suffices to show that $\varphi_a(u) = \varphi_{a^g}(u^g)$.

Consider the decomposition $u = \sum_{\lambda \in \mathcal{F}} u_\lambda$, where $u_\lambda \in A_\lambda(a)$. By applying g we get $u^g = \sum_{\lambda \in \mathcal{F}} u_\lambda^g$. On the other hand, decomposing u^g with respect to a^g, we get $u^g = \sum_{\lambda \in \mathcal{F}} v_\lambda$ where $v_\lambda \in A_\lambda(a^g)$. However, we have already observed that $A_\lambda(a^g) = A_\lambda(a)^g$. In particular, for $\lambda = 1$, we have

$$\varphi_a(u)a^g = (u_1)^g = v_1 = \varphi_{a^g}(u^g)a^g$$

Whence we see that $\varphi_{a^g}(u^g) = \varphi_a(u)$. □

Remark 4.6. Firstly, note that the algorithm in the proof of the second part has choice over the decompositions, so given w, there may be several ways of moving factors of w over to reduce it to an axis a. However, if $(\cdot, \cdot)$ is a Frobenius form, then any of these different ways of reducing it must give the same answer.

This shows that the value (a, a) on one axis may determine the value for other axes (b, b). So, not all choices of (a, a) for axes $a \in X$ lead to valid Frobenius forms. Indeed, if $\varphi_a(b) \neq 0$ for axes a and b, then

$$(a, a)\varphi_a(b) = (a, b) = \varphi_b(a)(b, b)$$

So, if $(a, a) \neq 0 \neq (b, b)$, then the value of (a, a) determines the value of (b, b).

As noted before, we often put restrictions on the values of (a, a) for axes A. In view of Proposition 4.5, we call the Frobenius form satisfying $(a, a) = 1$ for all generating axes a the *projection form*. This is what has also previously just been called the Frobenius form. We see from Proposition 4.5, that it is also invariant under the action of the Miyamoto group $\mathrm{Miy}(X)$.

It is not known whether an axial algebra always admits a Frobenius form. However, all currently known examples do. In particular, it has been shown that axial algebras of Jordan type η always admit a Frobenius form [12]. We also have the following conjecture:

Conjecture 4.7 (McInroy and Shpectorov)**.** *Assume that* $\mathrm{char}(\mathbb{F}) \neq 2$. *Then every primitive axial algebra of type* $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$ *admits a Frobenius form.*

Majorana algebras are the precursors of axial algebras introduced by Ivanov. As such, we can give a definition of them in terms of axial algebras.

Definition 4.8. A *Majorana algebra* is an axial algebra A of Monster type $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$ over $\mathbb{R}$ such that

M1 A has a (projection) Frobenius form $(\cdot,\cdot)$ which is positive definite.

M2 *Norton's inequality* holds. That is, for all $x, y \in A$,

$$(x^2, y^2) \geq (xy, xy)$$

In different papers, there are also additional axioms on the subalgebras assumed such as the M8 axiom.

5 Ideals

When we have an algebra, it is natural to study its ideals. Moreover, the quotients of axial algebras will often also be axial algebras, hence giving us new examples. The results from this section are largely due to Khasraw, McInroy and Shpectorov in [16].

Recall from Section 3.4, that if W is a subspace of an axial algebra A which is invariant under the action of ad_a, then

$$W = \bigoplus_{\lambda \in \mathcal{F}} W_\lambda$$

where $W_\lambda := W \cap A_\lambda$.

But an ideal I is invariant under the adjoint action by all elements. So in particular, for every $a \in X$,

$$I = \bigoplus_{\lambda \in \mathcal{F}} I_\lambda(a)$$

where $I_\lambda(a) := I \cap A_\lambda(a)$.

This led to us seeing earlier in Corollary 3.16 that ideals are invariant under the action of the Miyamoto group.

For the remainder of this section we will assume that our axial algebra is primitive. So, $I_1(a) \neq 0$ if and only if $a \in I$. This observation allows us to (usefully) split ideals into two categories:

- Ideals which do not contain any axes
- Ideals which do contain axes

5.1 The radical

Let us first consider ideals which do not contain any axes.

Lemma 5.1. *Let $Y \subseteq X$ be a set of primitive axes. Then there is a unique largest ideal that contains no axes in Y.*

Proof. By our observation, an ideal I does not contain an axis a if and only if $I_1(a) = 0$. That is, if and only if $I \subseteq A_{\mathcal{F}-\{1\}}(a)$. Hence an ideal I contains no axes from Y if and only if it is contained in $\bigcap_{a\in Y} A_{\mathcal{F}-\{1\}}(a)$. Clearly the sum of any two such ideals also lies in this intersection, so there is indeed a unique largest such ideal. $\square$

The following definition is now well-defined.

Definition 5.2. The *radical* $R(A, X)$ of a primitive axial algebra A is the (unique) largest ideal which does not contain any axes in X.

Since the definition of the radical requires primitivity, we will always assume this when talking abut the radical. A priori, the definition of the radical seems to depend on a choice of axes X. What happens if we change our choice of axes?

Theorem 5.3. *The radical is stable. That is, if $X \simeq Y$ are two equivalent sets of primitive axes, then $R(A, X) = R(A, Y)$.*

Proof. It is enough to show that $R(A, X) = R(A, \bar{X})$ for any generating set of axes X. Indeed, by Theorem 3.17, generation is stable and so the radical $R(A, Y)$ is defined. Hence, $R(A, X) = R(A, \bar{X}) = R(A, \bar{Y}) = R(A, Y)$.

By definition $R(A, \bar{X})$ does not contain any axes from $\bar{X}$. So in particular, it does not contain any axes from $X \subseteq \bar{X}$ and hence $R(A, \bar{X}) \subseteq R(A, X)$. On the other hand, every ideal is invariant under the Miyamoto group. So $R(A, X)$ also does not contain any axes from $X^{\mathrm{Miy}(X)} = \bar{X}$. That is, $R(A, X) \subseteq R(A, \bar{X})$ and hence $R(A, X) = R(A, \bar{X})$. $\square$

In light of the above, where X is clear from context, we may just write $R(A)$ for the radical.

Although the radical is a well-defined ideal, how would you try to calculate it? Being defined by the property that it is the largest ideal containing no axes does not make this easy. However, it will be easy when we have a Frobenius form.

Recall that for a bilinear form $(\cdot, \cdot)$ on A, we define the radical of the form to be

$$A^\perp = \{v : (v, a) = 0 \text{ for all } a \in A\}$$

Lemma 5.4. *The radical $A^\perp$ of a Frobenius form is an ideal of A.*

Proof. The radical $A^\perp$ is a subspace of A, so we need just show that it is closed under multiplication. Suppose $v \in A^\perp$ and $x \in A$. For all $y \in A$, $(vx, y) = (v, xy)$ and since $v \in A^\perp$, $(v, xy) = 0$. So $xv \in A^\perp$ and hence $A^\perp$ is closed under multiplication by $x \in A$ and so is an ideal. $\square$

Now we can show the following:

Theorem 5.5. *Let A be a primitive axial algebra with a Frobenius form $(\cdot, \cdot)$. Then the radical $R(A, X)$ equals the radical $A^\perp$ of the Frobenius form if and only if $(a, a) \neq 0$ for all $a \in X$.*

Proof. Let $R = R(A, X)$. Let $a \in X$. By Lemma 4.3, $\langle a \rangle \perp A_\lambda(a)$ for all $\lambda \in \mathcal{F}$ and so it is clear that $a \in A^\perp$ if and only if $(a, a) = 0$. In other words, $A^\perp$ contains no axes from X if and only if $(a, a) \neq 0$ for all $a \in X$. So if $R = A^\perp$, then R contains no axes from X.

Conversely, if $(a, a) \neq 0$, then $A^\perp \subseteq R$. It remains to show that $R \subseteq A^\perp$. Since X generates A, the algebra is spanned by products w of axes in X. We need to show that R is orthogonal to each product w, which we will do by induction on the length of the product w.

If the length of w is 1, then $w = a$ is an axis in X. Decomposing the ideal R with respect to a, we see that $R \subseteq A_{\mathcal{F}-\{1\}}(a)$ as a is primitive. Now again by Lemma 4.3, a is orthogonal to $A_{\mathcal{F}-\{1\}}(a)$ and hence $w = a$ is orthogonal to R as required.

Now suppose that w has length at least two, say $w = w_1 w_2$, where w_1 and w_2 have length strictly shorter than w. Now, $(w, R) = (w_1 w_2, R) = (w_1, w_2 R) = (w_1, R)$ as R is an ideal. However, by induction $0 = (w_1, R)$. So $(w, R) = 0$ as required and hence $R \subseteq A^\perp$ and therefore $R = A^\perp$. $\square$

We have already noted that all Jordan type algebras admit a Frobenius form and conjectured that all axial algebras of Monster type $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$ admit such a form. In fact, we do not know of any examples for any fusion law which do not admit a Frobenius form and moreover one where $(a, a) \neq 0$ for all $a \in X$.

So the above theorem is a powerful way for calculating the radical of an axial algebra. In many cases the radical will be trivial, or small, however there are algebras where the radical is very large. Indeed, there are algebras where it has codimension 1. For example, the highwater algebra [7], its characteristic 5 cover [6], and some of the families described by Yabe [25]. All such interesting examples of this are baric (a baric algebra is one where there is a homomorphism $\omega \colon A \to \mathbb{F}$; the kernel of ω is necessarily a codimension 1 ideal). In such cases, every though we can easily find the radical, it may be very difficult to identify all the ideals properly contained in the radical.

Exercise 5.6. For the Matsuo algebra $3\mathrm{C}(\eta) = M_\eta(S_3)$ defined from the group S_3 considered in Exercise 2.8, find the radical for all values of η.

5.2 The projection graph

We now turn our attention to the other class of ideals, namely ideals I which do contain an axis a. What other axes $b \in X$ are contained in I?

Recall that we can decompose a with respect to b and get $a = \sum_{\lambda \in cF} a_\lambda$, where $a_\lambda \in A_\lambda(b)$. If b is a primitive axis, then a_1 is a scalar multiple of b. In Section 4, we defined the projection map $\varphi_b \colon A \to \mathbb{F}$ by taking $\varphi_b(a)$ to be the scalar such that $a_1 = \varphi_b(a)b$.

By restricting the projection map to axes, we get a directed graph.

Definition 5.7. Let A be a primitive axial algebra. The *projection graph* Γ is the directed graph with vertex set X and a directed edge from a to b if the projection $\varphi_b(a)$ of a onto b is non-zero.

Given a directed graph, we can define the *out set* $\mathrm{Out}(\Gamma, Y)$ of a subset of vertices Y to be the set of all vertices reachable by a directed path from a vertex in Y.

Lemma 5.8. *Let A be a primitive axial algebra and Γ be its projection graph. If Y is a set of axes contained in an ideal I, then the axes in $\mathrm{Out}(\Gamma, Y)$ are also contained in I.*

Proof. Let $a, b \in X$ and write $a = \sum_{\lambda \in \mathcal{F}} a_\lambda$, where $a_\lambda \in A_\lambda(b)$. Suppose that $a \in I$ and there is a directed edge from a to b in Γ. That is, $\varphi_b(a) \neq 0$ and hence $a_1 \neq 0$. Since $a \in I$, we have $ba = b \sum_{\lambda \in \mathcal{F}} a_\lambda = \sum_{\lambda \in \mathcal{F}} \lambda a_\lambda \in I$. By continued multiplication by b, we see that $\sum_{\lambda \in \mathcal{F}} \lambda^k a_\lambda \in I$, for all $k \in \mathbb{N}$. Now, by taking linear combinations, we see that $a_\lambda \in I$, for all $\lambda \in \mathcal{F}$. In particular, $0 \neq a_1 \in I$. However, since A is primitive, since $\varphi_b(a) \neq 0$, $b = \frac{1}{\varphi_b(a)} a_1 \in I$. Now the result follows by transitivity of paths. $\square$

So the projection graph controls which axes are in an ideal. In fact we can do better by combining this with the fact that ideals are invariant under the Miyamoto group (Corollary 3.16). If X is a closed set of axes, then we can quotient Γ by the action of the Miyamoto group $G = \mathrm{Miy}(X)$ to get the *orbit projection graph* $\bar{\Gamma} := \Gamma/G$. Its vertices are orbits a^G and there is a directed edge from a^G to b^G if there exists axes $c \in a^G$ and $d \in b^G$ such that there is a directed edge from b to c in Γ. Now, we just need to check paths in the orbit projection graph.

For the first class of ideals where they contained no axes, it was much easier to find the radical when the algebra admitted a Frobenius form.

Lemma 5.9. *Let A be a primitive axial algebra with a Frobenius form $(\cdot, \cdot)$ and Γ be its projection graph. Suppose that $(a, a) \neq 0 \neq (b, b)$ for $a, b \in X$. Then the following are equivalent:*

1. $(a, b) \neq 0$

2. *There is a directed edge* $a \leftarrow b$ *in* Γ
3. *There is a directed edge* $a \rightarrow b$ *in* Γ

Proof. By Proposition 4.5(1), $(a, b) = \varphi_b(a)(a, a) = \varphi_a(b)(b, b)$. $\square$

So when we have a Frobenius form, we can consider the projection graph to be an undirected graph. In particular, the axes reachable from a given axes are those in the connected component of that axis.

Corollary 5.10. *Let* A *be a primitive axial algebra with a Frobenius form* $(\cdot, \cdot)$ *such that* $(a, a) \neq 0$ *for all* $a \in X$*. If its projection graph* Γ *is connected, then every ideal of* A *is contained in the radical.*

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